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On the plane problem of the flow around a submerged beam

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Abstract

We consider the problem of the steady flow of an ideal heavy fluid around a submerged beam. The problem is obtained from the free-boundary problem of the flow past a submerged obstacle in the limit of bodies of vanishing thickness. We introduce a special Sobolev space formulation of the problem in term of a perturbed stream function and prove its unique solvability for every value of the unperturbed flow velocity, with the possible exception of a discrete set depending on the geometry of the domain. The asymptotic properties of the solutions are discussed.

Keywords: Steady flow; Linear water waves; Neumann–Kelvin problem

1. Introduction

A classical problem in hydrodynamics is the study of the plane stationary flow of a heavy fluid over submerged obstacles. Assuming the usual hypotheses, i.e., irrotational and divergence-free flow, nonviscous fluid and negligible surface tension, the velocity field can be described by a complex function, holomorphic in the domain occupied by the fluid; such a domain has rigid boundaries, where the "no-flow condition" is imposed, and a free boundary where, in addition, the nonlinear dynamical condition (Bernoulli condition) holds. When no obstacle is contained in the fluid, the only rigid boundary is a horizontal bottom and we have the well-known steady water-wave problem.

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Although substantial progress has been achieved in the rigorous treatment of the latter problem (even in the three-dimensional case) [1], little is known about the interaction of water waves with submerged or semi-submerged rigid bodies. There are some results concerning steady flows under localized pressure perturbations on the free surface, which may simulate the action of a surface-piercing obstacle [2,3]; more recently, exact solvability has been proved for the complementary physical problem of the *ship waves* generated by the uniform horizontal motion of a *thin semi-submerged body* [4]. In the case of a completely submerged cylinder, exact solutions have been found in [5] only for supercritical flow velocities (see below).

Nevertheless, most of the mathematical treatment of the problem deals with linearized versions, whose solutions are called *linear water waves* [6]. Even in dimension two, one of the still open questions in the linear theory is to determine whether the problem for a given obstacle in a current is uniquely solvable for all values of the flux velocity. A positive answer is known for special geometries [7,8], but there are examples of nontrivial, finite energy solutions of the homogeneous problem (trapped modes) in the presence of multiple obstacles and in the case of a submerged hollow [9]. In general, the connection between unique solvability and the geometry of the obstacles is not completely understood. The standard approach to solvability of the linear problem relies on integral equation techniques [6], but recently an alternative variational treatment has been proposed, which is suitable for a large set of obstacles (particularly when the perturbed flow has finite energy [10]) including the limit case of a *surface beam* [11,12].

The aim of this paper is to apply the same technique to the problem for a *submerged horizontal beam*, which cannot be treated by standard integral equation methods as in the case of a body with smooth boundary [13]. The interest in this problem is that it is obtained from the exact, free-boundary problem with an obstacle of thickness of order ϵ, in the limit $\epsilon \to 0$; thus, the study of this limit case is a crucial step in treating the nonlinear problem of the flow past a *thin submerged body* by local methods, following the same pattern which led us to prove the solvability of the analogous problem for a surface-piercing body [4]. The linear problem, usually formulated for a velocity potential, is stated in terms of a *stream function*, which satisfies more regular boundary conditions. In particular, we get on the free surface the same *Steklov condition* previously obtained for the vertical component of the velocity field in the problem for a surface beam both in a supercritical [11] and subcritical stream [12]; in the case of a submerged beam, however, we cannot state the problem in terms of a single component of the velocity field since the domain is not simply connected.

The problem depends on the (unperturbed) flow velocity c through the positive parameter $\nu = g/c^2$, where g is the acceleration of gravity. The main conclusion is that unique solvability holds for every ν with the possible exception of a sequence of values $\nu^{(n)} > 1/b$, where b is the depth of the submerged beam. Then, for every value of c with the possible exception of a discrete set in the interval $0 < c < \sqrt{gb}$, we get a one-parameter family of solutions for the velocity field; a particular solution can be chosen either by prescribing the value of circulation around the obstacle or by selecting a “least singular solution” (see Section 2 below).

2. The problem

We consider a horizontal beam contained in a strip

$$\Sigma \equiv \{x \in \mathbb{R};\ -H < y < 0\},$$

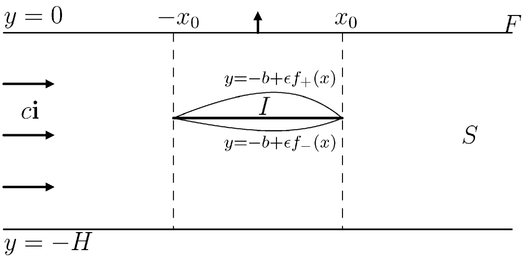

Fig. 1.

whose upper and lower boundary consists of a plane surface $F = \mathbb{R} \times \{0\}$ (which will be improperly called free boundary) and of a flat bottom $B = \mathbb{R} \times \{-H\}$; the beam is represented by a segment $I = (-x_0, x_0) \times \{-b\}$, where $x_0 > 0$ and $0 < b < H$; finally, we define the *strip with a cut*

$$S = \Sigma \setminus I.$$

As mentioned above, we can consider the beam I as the limit, for $\epsilon \to 0$, of a thin, rigid body represented by the domain

$$\big\{(x, y) \in \Sigma\colon -x_0 < x < x_0,\ -b + \epsilon f_-(x) < y < -b + \epsilon f_+(x)\big\},$$

where $f_\pm \in \mathcal{C}^1([-x_0, x_0])$ (see Fig. 1).

We assume that the body is surrounded by a steady flow with (unperturbed) field velocity $c\mathbf{i}$ at upstream infinity and we denote by $\epsilon(u\mathbf{i} + v\mathbf{j})$ the perturbed velocity field; moreover, for a zero circulation flow one can define a potential ϕ such that $\nabla\phi = u\mathbf{i} + v\mathbf{j}$. Then, we obtain in the limit $\epsilon \to 0$, the following linear system (see [5]):

$$\Delta\phi = 0 \quad \text{in } S, \tag{2.1}$$

$$\phi_{xx} + \nu\phi_y = 0 \quad \text{on } F, \tag{2.2}$$

$$\partial_y\phi_\pm = K_\pm \quad \text{on } I, \tag{2.3}$$

$$\phi_y = 0 \quad \text{on } B, \tag{2.4}$$

where $\partial_y\phi_\pm$ are the limit values of $\partial_y\phi = v$, respectively on the upper and lower side of the beam and

$$K_\pm = cf'_\pm \tag{2.5}$$

by the no-flow condition. The problem is completed by the asymptotic conditions

$$\lim_{x \to -\infty} \nabla\phi(x, y) = 0 \quad (\text{uniformly in } y), \tag{2.6}$$

$$\sup_{(x,y) \in S \setminus A} \big|\nabla\phi(x, y)\big| < \infty, \tag{2.7}$$

where A is any neighborhood of I. The problem (2.1)–(2.4) (for any $\nu > 0$ and regular enough $K_\pm$) with the asymptotic conditions (2.6), (2.7), is called the *Neumann–Kelvin problem* (for a beam); this problem has been widely studied in the literature, since its solution also represents the potential of the *linear ship waves* generated by a body moving in a fluid with uniform motion [6]. However, in two dimensions one can provide an alternative description of the problem by considering the *stream function* ψ (harmonic conjugate of the potential ϕ) satisfying $\psi_x = v$, $-\psi_y = u$. We recall that the function ψ is determined (except for an arbitrary constant) by the velocity field of an incompressible fluid (such that $u_x + v_y = 0$) whenever the *flux* $\oint_{\mathcal{C}}\{v\,dx - u\,dy\}$ through a closed curve $\mathcal{C}$ surrounding the obstacle is vanishing. In the case of a beam, by shrinking $\mathcal{C}$ to a segment we get

$$\int\limits_I (v_+ - v_-)\,dx = 0, \tag{2.8}$$

where $v_\pm$ are the limit values of v respectively on the upper and lower side of the beam; as expected, (2.8) holds for a solution satisfying the "no-flow" condition (i.e. with boundary data of the form (2.5), see below). It follows that the stream function can be defined also in the *more general* case of flows with nonzero circulation around the obstacle. We also remark that the stream function was recently employed to provide new variational formulations for nonlinear steady water waves with vorticity [14]. Moreover, as we now show, the linear problem for the function ψ has the advantage that the second order boundary condition (for the potential) on the free surface is replaced by a first order Steklov condition. For, by the definition of ψ and observing that both F and B are *connected sets*, it follows by (2.2) that $\psi_y - \nu\psi$ is constant on F and by (2.4) that ψ is constant on B; by suitably choosing the arbitrary constant in the definition of ψ, we may assume $\psi = 0$ on B. Then, since $\lim_{x\to-\infty}\psi_y = 0$ (by (2.6)) we also get $\psi \to 0$ for $x \to -\infty$ (uniformly in y) so that $\psi_y - \nu\psi = 0$ on F. Finally, on the two sides of the beam we may assign to ψ a pair of *Dirichlet boundary data* which correspond to condition (2.3) (see below). Summing up the discussion, we have the problem:

Problem P_ν. For every $\nu > 0$, find ψ satisfying

$$\Delta\psi = 0 \quad \text{in } S, \tag{2.9}$$

$$\psi_\pm = \mathcal{K}_\pm \quad \text{on } I, \tag{2.10}$$

$$\psi_y - \nu\psi = 0 \quad \text{on } F, \tag{2.11}$$

$$\psi = 0 \quad \text{on } B, \tag{2.12}$$

$$\lim_{x\to-\infty} \nabla\psi(x,y) = 0 \quad \text{(uniformly in } y\text{)}, \tag{2.13}$$

$$\sup_{(x,y)\in S\setminus A} \left|\nabla\psi(x,y)\right| < \infty, \tag{2.14}$$

where A is any neighborhood of I. By the definition of ψ, the boundary data $\mathcal{K}_\pm$ in condition (2.10) have to satisfy $\mathcal{K}'_\pm = v_\pm$, so that by (2.8) we obtain (for absolutely continuous $\mathcal{K}_\pm$)

$$(\mathcal{K}_+ - \mathcal{K}_-)(x_0) = (\mathcal{K}_+ - \mathcal{K}_-)(-x_0). \tag{2.15}$$

Assume now, according with (2.3), (2.5), that $v_\pm = cf'_\pm$ (the datum of the linearized flow problem); then, we get $\mathcal{K}_\pm = cf_\pm + \gamma_\pm$, where $\gamma_\pm$ are arbitrary constants (note that (2.15) is satisfied since the obstacle's edge is a closed curve). However, if we look for regular enough (H^1) solutions in S, we are forced to take

$$\mathcal{K}_\pm = cf_\pm + \gamma, \tag{2.16}$$

with γ arbitrary constant (see Eq. (3.4) below). We still need an additional condition on the linearized flow in order to fix γ; this is not a surprise since in stating problem (2.1)–(2.7) for the potential field we made the extra assumption that the flow has vanishing circulation. In our case the circulation

$$\oint_{\mathcal{C}} \{u\,dx + v\,dy\} = \int_I \big\{(\partial_y \psi)_+ - (\partial_y \psi)_-\big\}\,dx \tag{2.17}$$

is a linear functional of ψ, so that γ could be determined by prescribing the value of (2.17) on the solutions with Dirichlet data (2.16) (see Section 8).

An alternative condition, which is of local type and is common in studying fluid flows (both in hydrodynamics and in aerodynamics) is related to the determination of the *least singular solution* (see e.g. [6, Section 8.3]). In fact, it is known that in general the velocity field is singular near beam tips and it is relevant to investigate the case when the singularity is absent or it appears only at one point; as we will show in Section 8, this additional condition fixes the value of the parameter γ.

3. Variational formulation for finite energy solutions

We are concerned with unique solvability of problem P_ν for any value of the positive parameter ν. As it is known, a critical value of this parameter is $\nu = 1/H$; in fact, when $\nu < 1/H$ (supercritical flow) it can be proved that the problem is uniquely solvable and that every solution is exponentially vanishing at infinity (this corresponds to the case of a solitary wave in the water-wave problem). This result relies on a variational formulation of the problem, which can also applied to other types of obstacles in a supercritical stream [10]; on the contrary, there are no results on the resolubility of the problem for $\nu > 1/H$ (subcritical flow). In fact, the standard methods mentioned in the introduction, which (partially) solve the problem in the case of smooth obstacles, are not suitable for a beam; thus, it seems reasonable to consider the extension of the variational approach to the subcritical flow. In this case, however, the perturbed flow will not vanish (in general) at infinity downstream, since steady periodic water waves are known to exist in the subcritical regime; therefore, the solution will not belong to a functional space with finite Dirichlet norm. Nevertheless, as it was shown in the case of semi-submerged obstacles [8], one can search *weak solutions of finite energy* belonging to some subspace of the Sobolev space $H^1(S)$; subsequently, by a suitable regularization procedure, one uniquely associates to a given variational solution a solution of problem P_ν. In this section, we introduce the variational formulation of the problem and discuss its solvability on suitable subsets of $H^1(S)$. As we will see, the value $\nu = 1/b$ (where b is the depth of the beam) is also critical in the proof of solvability. To begin with, we take into account condition (2.12) by defining the subspace of the functions with vanishing trace on B:

$$H^1_B(S) := \big\{\psi \in H^1(S),\ \psi|_B = 0\big\}. \tag{3.1}$$

We now show that the functions in $H^1_B(S)$ satisfy the Poincarè inequality; hence, the Sobolev norm on H^1_B is equivalent to the Dirichlet norm:

$$\|\psi\|^2 = \int_S |\nabla\psi|^2\,dx\,dy. \tag{3.2}$$

Proposition 3.1. *For any $\psi \in H^1_B(S)$ there holds*

$$\int_S |\psi|^2\,dx\,dy \leqslant C\int_S |\nabla\psi|^2\,dx\,dy, \tag{3.3}$$

for some positive constant C.

Proof. Fix $R > x_0$ and consider the restrictions of ψ to

$$S_R = S \cap \big\{(x, y),\ |x| < R\big\}$$

and to $S \setminus S_R$. In the latter (unbounded) region the bound

$$\int_{S\setminus S_R} |\psi|^2\,dx\,dy \leqslant C \int_{S\setminus S_R} |\nabla\psi|^2\,dx\,dy$$

is directly verified by elementary estimates on smooth functions vanishing at the edge of a strip and by standard density arguments; moreover, we can divide S_R into two *bounded and Lipschitz* domains (e.g., the rectangle $R_0 = (-x_0, x_0) \times (-H, -b)$ and its complement in S_R) such that ψ vanishes on some part of their boundaries with positive one-dimensional measure. Since the Poincaré inequality holds for such functions and domains [15, Chapter 4, §7], the bound (3.3) follows. $\square$

In order to properly define a weak formulation of the problem in $H^1_B(S)$, we must take into account the trace properties of H^1 functions in a domain with a cut. In fact, if we look for solutions in $H^1(S)$ satisfying the boundary condition (2.10), it is necessary to assume $\mathcal{K}_\pm \in H^{1/2}(I)$, together with certain *compatibility conditions* for the data in the neighborhood of the end points of the beam $(\pm x_0, -b)$ [16]; in the case of continuous $\mathcal{K}_\pm$, these conditions simply become

$$\mathcal{K}_+(\pm x_0) = \mathcal{K}_-(\pm x_0), \tag{3.4}$$

which obviously imply the previous zero flux condition (2.15). Note further that assuming the no-flow condition $v_\pm = cf'_\pm$ and recalling that $f_+(x_0) = f_-(x_0)$ and $f_+(-x_0) = f_-(-x_0)$, condition (3.4) forces the Dirichlet data (2.16) in Eq. (2.10). We can now state the *variational form of problem P_ν*:

Find $\psi \in H^1_B(S)$ satisfying $\psi_\pm = \mathcal{K}_\pm$ on the two sides of I and such that

$$\int_S \nabla\psi\nabla v\,dx\,dy - \nu\int_F \psi v\,dx = 0, \tag{3.5}$$

for every $v \in H^1_B(S)$, with $v_\pm = 0$.

By our previous assumptions on the boundary data, there exists $\psi_0 \in H^1_B$ satisfying the boundary conditions (2.10); then, we can write $\psi = \psi_1 + \psi_0$, where ψ_1 vanishes on I and satisfies

$$\int_S \nabla\psi_1 \nabla v \, dx\, dy - \nu \int_F \psi_1 v \, dx = -\int_S \nabla\psi_0 \nabla v \, dx\, dy + \nu \int_F \psi_0 v \, dx, \tag{3.6}$$

for every $v \in H^1_B(S)$, with $v_\pm = 0$. It is readily verified that the bilinear form and the linear functional in Eq. (3.6) are continuous on $H^1_B(S)$ for every value of the parameter ν; on the contrary, *coercivity depends critically on* ν. To see this in detail, let us define as before

$$S_{x_0} = S \cap \bigl\{(x, y),\ |x| < x_0\bigr\},$$

and let $Q_0 = (-x_0, x_0) \times (-b, 0)$; note that $S_{x_0} = R_0 \cup Q_0$, where R_0 was defined in the proof of Proposition 7.2.

Then, by standard calculations using Hölder inequality [9,11], we have

$$\int_{S \setminus S_{x_0}} |\nabla v|^2 \, dx\, dy - \nu \int_{F \setminus [-x_0, x_0]} |v|^2 \, dx \geqslant (1 - \nu H) \int_{S \setminus S_{x_0}} |\nabla v|^2 \, dx\, dy. \tag{3.7}$$

Similarly, *if* v *vanishes on the upper side of* I:

$$\int_{Q_0} |\nabla v|^2 \, dx\, dy - \nu \int_{-x_0}^{x_0} |v|^2 \, dx \geqslant (1 - \nu b) \int_{Q_0} |\nabla v|^2 \, dx\, dy. \tag{3.8}$$

By the previous estimates, we are led to consider three disjoint intervals of values for the parameter ν: $0 < \nu < 1/H$, $1/H < \nu < 1/b$ and $\nu > 1/b$.

Actually, if ν belongs to the first interval, we get $1 - \nu b > 1 - \nu H > 0$ in (3.7), (3.8), so that we can use both the estimates to achieve coercivity of the bilinear form (3.6). In the second interval we have $1 - \nu H < 0$, so that we loose the coercivity estimate in $S \setminus S_{x_0}$; finally, in the last interval both the coefficients at the right-hand sides of (3.7), (3.8) become negative.

As previously remarked, when $\nu < 1/H$ one easily gets (by Lax–Milgram theorem) a unique variational solution $\psi \in H^1_B(S)$; then, by standard methods, one shows that ψ is harmonic in S, smooth outside any neighborhood of the beam and exponentially vanishing at infinity [12]. We will now focus on the remaining two cases.

4. Weak solutions for $1/H < \nu < 1/b$

We first remark that for every $\nu > 1/H$ there are two independent solutions of the "free problem," i.e., $\Delta\psi = 0$ in the strip Σ defined in Section 2, $\psi = 0$ on B and $\psi_y = \nu\psi$ on F; they are given by

$$S_0(x, y) = \sin(\nu_0 x) \sinh\bigl(\nu_0(y + H)\bigr), \qquad C_0(x, y) = \cos(\nu_0 x) \sinh\bigl(\nu_0(y + H)\bigr), \tag{4.1}$$

where ν_0 is the positive solution of

$$\tanh(\nu_0 H) = \frac{\nu_0}{\nu}. \tag{4.2}$$

It is convenient to define, for a fixed $\xi \in \mathbb{R}$, two linear combinations of these functions, namely

$$\begin{aligned} \Phi_0(x, y; \xi) &= \cos(\nu_0 \xi) S_0(x, y) + \sin(\nu_0 \xi) C_0(x, y), \\ \Psi_0(x, y; \xi) &= \cos(\nu_0 \xi) S_0(x, y) - \sin(\nu_0 \xi) C_0(x, y). \end{aligned} \tag{4.3}$$

By applying Green's formula to ψ and to the second of (4.3) in the half strips not including the beam we get the identity (see [8, §2.1])

$$\int_{-H}^{0} \sinh\bigl[\nu_0(y + H)\bigr]\psi(\xi, y)\, dy = 0, \quad \forall |\xi| \geqslant x_0, \tag{4.4}$$

which by integration by parts can be written

$$\cosh(\nu_0 H)\psi(x, 0) = \int_{-H}^{0} \cosh \nu_0(y + H)\psi_y(\xi, y)\, dy. \tag{4.5}$$

By squaring (4.5) and integrating on F we readily get [8]

$$\nu \int_{F\setminus[-x_0, x_0]} |\psi|^2 \leqslant \frac{1}{2}\left(1 + \frac{2\nu_0 H}{\sinh(2\nu_0 H)}\right)\|\nabla\psi\|^2_{L^2(S\setminus S_{x_0})}. \tag{4.6}$$

Notice that the constant appearing at the right-hand side is *strictly less than* 1 for every $\nu_0 > 0$. Formula (4.6) and the form (3.6) of the variational problem suggest the definition of the following (closed) subspace of $H^1_B(S)$

$$U_* = \left\{\psi \in H^1_B(S)\colon\ \psi|_I = 0,\ \int_{-H}^{0} \sinh\bigl[\nu_0(y + H)\bigr]\psi(x, y)\, dy = 0\ \forall |x| \geqslant x_0\right\}. \tag{4.7}$$

Now, the bound (4.6) can be exploited to obtain a *new coercivity estimate on* U_* which replaces (3.7). Then, one can prove the following

Theorem 4.1. *Let* $1/H < \nu < 1/b$ *and let* $\mathcal{K}_\pm \in H^{1/2}(I)$ *be continuous functions satisfying* (3.4). *Then, there is* $\psi \in H^1_B(S)$ *which satisfies the boundary conditions* (2.10), (2.12) (*in the sense of the traces of* H^1 *functions*) *and such that* (3.5) *holds* $\forall v \in U_*$. *Moreover,* ψ *is uniquely determined if one requires the additional condition* (4.4).

Proof. We write $\psi = \psi_1 + \psi_0$ as before Eq. (3.6), where $\psi_0 \in H^1_B$ satisfies the boundary conditions (2.10) and $\psi_1|_I = 0$. One can show that ψ_0 can be chosen to satisfy (4.4) (see the proof of Theorem 5.1 below). Then, the problem is reduced to find $\psi_1 \in U_*$ satisfying (3.6) for every $v \in U_*$. We claim that the bilinear form at the left-hand side of (3.6) is coercive on U_*; in fact, every $v \in U_*$ satisfies (4.6), so that, by (3.8) and by the definition (3.2) of the norm, we get

$$\int_S |\nabla v|^2\,dx\,dy - \nu \int_F |v|^2\,dx$$

$$\geqslant \frac{1}{2}\left(1 - \frac{2\nu_0 H}{\sinh(2\nu_0 H)}\right)\|\nabla v\|^2_{L^2(S\setminus S x_0)} + (1-\nu b)\|\nabla v\|^2_{L^2(S x_0)} \geqslant \alpha\|v\|^2, \tag{4.8}$$

where $\alpha = \min\{1-\nu b, \frac{1}{2}(1 - \frac{2\nu_0 H}{\sinh(2\nu_0 H)})\} > 0$. Hence, our claim follows. Then, (3.6) has the unique solution $\psi_1 \in U_*$, so that $\psi = \psi_1 + \psi_0$ satisfies (3.5) (for every $v \in U_*$), (4.4) and the boundary conditions. It remains to prove uniqueness; if $\hat{\psi}$ is another solution satisfying (4.4) and the same boundary conditions as ψ, it follows that $\psi - \hat{\psi}$ is a solution in U_* of the variational equation (3.5), so that $\psi - \hat{\psi} = 0$ by coercivity. □

5. Weak solutions for $\nu > 1/b$

In this case, we have to introduce further restrictions on the test functions in the variational equation in order to achieve an additional coercivity estimate for $|x| \leqslant x_0$ similar to (4.8). To this aim, we proceed as in the previous section and consider an *a priori relation* valid for any harmonic function ψ in the rectangle $Q_0 = (-x_0, x_0) \times (-b, 0)$, which satisfies (2.10) and (2.11).

Let us now define the functions

$$S_1(x,y) = \sin(\nu_1 x)\sinh\bigl(\nu_1(y+b)\bigr), \qquad C_1(x,y) = \cos(\nu_1 x)\sinh\bigl(\nu_1(y+b)\bigr), \tag{5.1}$$

where ν_1 is the positive solution of

$$\tanh(\nu_1 b) = \frac{\nu_1}{\nu}. \tag{5.2}$$

As before, we define the two linear combinations

$$\begin{aligned}
\Phi_1(x,y;\xi) &= \cos(\nu_1\xi)S_1(x,y) + \sin(\nu_1\xi)C_1(x,y),\\
\Psi_1(x,y;\xi) &= \cos(\nu_1\xi)S_1(x,y) - \sin(\nu_1\xi)C_1(x,y).
\end{aligned} \tag{5.3}$$

Then, for any fixed point $(\xi, 0) \in F$, with $|\xi| \leqslant x_0$ we apply Green's theorem to ψ and Ψ_1 in the rectangle $(\xi, x_0) \times (-b, 0)$; by elementary calculations we get (see also [9, Lemma 2.3])

$$\int_{-b}^{0} \psi(\xi,y)\sinh\bigl[\nu_1(y+b)\bigr]\,dy = \mathcal{H}(\xi), \tag{5.4}$$

where

$$\mathcal{H}(\xi) = \int_{\xi}^{x_0} \mathcal{K}_+(x)\sin\bigl[\nu_1(\xi - x)\bigr]\,dx + \alpha\cos(\nu_1\xi) + \beta\sin(\nu_1\xi) \tag{5.5}$$

and α, β are suitable constants.

Now, for any given $\mathcal{H} \in H^1(-x_0, x_0)$ we define the subset $\mathcal{W}_{\mathcal{H}} \subset H^1_B$ of the functions ψ satisfying (5.4) for every $\xi \in [-x_0, x_0]$ and (4.4) for $|\xi| \geqslant x_0$. Note that, being the intersection of closed hyperplanes in H^1_B, $\mathcal{W}_{\mathcal{H}}$ is a *closed and convex subset*. If $\mathcal{H}=0$, the homogeneous conditions (5.4) and (4.4) define a *subspace* $\mathcal{W}_0$; we further define the *subspace*

$$V_* = \{v \in \mathcal{W}_0,\ v|_I = 0\}. \tag{5.6}$$

Clearly, we have $V_* \subset U_*$ (see (4.7)). We can now state

Theorem 5.1. *Let $\nu > 1/b$ and let $\mathcal{K}_\pm \in H^{1/2}(I)$ be continuous functions satisfying* (3.4)*. Then, for every given $\mathcal{H} \in H^1(-x_0, x_0)$, there is a unique $\psi \in \mathcal{W}_{\mathcal{H}}$ satisfying the boundary conditions* (2.10)*,* (2.12) (*in the sense of the traces of H^1 functions*) *and such that* (3.5) *holds* $\forall v \in V_*$.

Proof. We write again $\psi = \psi_1 + \psi_0$, where $\psi_0 \in H^1_B$ satisfies the boundary conditions (2.10) and $\psi_1|_I = 0$. We now show that we can choose $\psi_0 \in \mathcal{W}_{\mathcal{H}}$. Let χ_0, χ_1, be smooth functions with $\operatorname{supp}\chi_0 \in (-H, -b)$, $\operatorname{supp}\chi_1 \in (-b, 0)$ and such that

$$\int \sinh\bigl[\nu_0(y+H)\bigr]\chi_0(y)\,dy = \int \sinh\bigl[\nu_1(y+b)\bigr]\chi_1(y)\,dy = 1,$$

$$\int \sinh\bigl[\nu_0(y+H)\bigr]\chi_1(y)\,dy = 0,$$

where ν_0, ν_1 satisfy respectively (4.2) and (5.2); moreover, let $\tilde{\mathcal{H}} \in H^1(\mathbb{R})$ be any extension of $\mathcal{H}$. Then, for every $\psi_0 \in H^1_B$ we define

$$\begin{aligned}\tilde\psi_0(x,y) = \psi_0(x,y) &- \left[\int_{-H}^{0} \psi_0(x,s)\sinh\bigl[\nu_0(s+H)\bigr]\,ds\right]\chi_0(y)\\ &+ \left[\tilde{\mathcal{H}}(x) - \int_{-b}^{0} \psi_0(x,s)\sinh\bigl[\nu_1(s+b)\bigr]\,ds\right]\chi_1(y).\end{aligned}$$

It is readily verified that $\tilde\psi_0$ belongs to $\mathcal{W}_{\mathcal{H}}$ and satisfies the same boundary conditions as ψ_0 (in the sequel, we still denote by ψ_0 such a function).

We now have to find $\psi_1 \in V_*$ satisfying (3.6) for every $v \in V_*$. By recalling the discussion of the previous section, it is straightforward to check the coercivity of the left-hand side of (3.6) in V_*; actually, by the same calculations that led from (4.4) to (4.6), we also get

$$\nu \int_{-x_0}^{x_0} |v|^2\,dx \leqslant \frac{1}{2}\left(1 + \frac{2\nu_1 b}{\sinh(2\nu_1 b)}\right)\|\nabla v\|^2_{L^2(Q_0)},$$

for every $v \in V_*$. By this bound and by (4.6) we now have

$$\begin{aligned}&\int_S |\nabla v|^2\,dx\,dy - \nu\int_F |v|^2\,dx\\ &\quad\geqslant \frac{1}{2}\left(1 - \frac{2\nu_0 H}{\sinh(2\nu_0 H)}\right)\|\nabla v\|^2_{L^2(S\setminus S x_0)} + \frac{1}{2}\left(1 - \frac{2\nu_1 b}{\sinh(2\nu_1 b)}\right)\|\nabla v\|^2_{L^2(S x_0)}\end{aligned} \tag{5.7}$$

and we conclude that coercivity holds with $\alpha = \frac{1}{2}(1 - \frac{2\nu_1 b}{\sinh(2\nu_1 H_1)}) > 0$. (Note that, by (4.2) and (5.2) we have $\nu_0 H > \nu_1 b$.) Then, (3.6) is uniquely solvable in V_*, so that $\psi = \psi_1 + \psi_0 \in \mathcal{W}_{\mathcal{H}}$ satisfies (3.5) for every $v \in V_*$ and the boundary conditions. Uniqueness follows as in Theorem 4.1. □

Remark 5.2. We stress that, in order to uniquely determine the weak solution ψ of the previous theorem, *we need a supplementary condition* specified by the function $\mathcal{H}$ in (5.4); clearly, if we require that ψ is harmonic in Q_0, it is necessary to choose $\mathcal{H}$ as in (5.5). However, even in this case we are left with two undetermined constants α, β. In particular, we have a two-dimensional subspace of weak solutions of the problem with homogeneous boundary data (see below).

6. Properties of the variational solutions

We now investigate the properties of the solutions given by Theorems 4.1 and 5.1 of the previous sections. As we will see, a weak solution is represented by a function which (in general) is not harmonic in the domain S; this is because, as we already stressed, a solution of problem P_ν with $\nu > 1/H$ does not vanish at downstream infinity. The subsequent theorems characterize the variational solutions in the two regimes $1/H < \nu < 1/b$ and $\nu > 1/b$.

Theorem 6.1. *Assume that $1/H < \nu < 1/b$ and let $\psi \in H^1_B(S)$ be defined by Theorem* 4.1. *Then, ψ satisfies the boundary conditions of problem P_ν and there exist real constants λ_+, λ_- such that*

$$\Delta\psi(x, y) = \left[\lambda_+\delta(x - x_0) + \lambda_-\delta(x + x_0)\right]\sinh\left[\nu_0(y + H)\right], \tag{6.1}$$

for $(x, y) \in S$, where δ denotes the Dirac delta function. Moreover, if $\hat{\psi} \in H^1_B(S)$ solves (6.1) *with (possibly different) constants $\hat{\lambda}_\pm$ and satisfies the same boundary conditions as ψ, then $\hat{\psi} = \psi$ (and therefore $\hat{\lambda}_\pm = \lambda_\pm$).*

Proof. Let φ_0 be a smooth function with support in S and such that

$$\int_{-H}^{0} \varphi_0(\pm x_0, y)\sinh\left[\nu_0(y + H)\right]dy = 0. \tag{6.2}$$

Furthermore, we set

$$\phi_0(x) = c_0^{-1}\int_{-H}^{0} \varphi_0(x, y)\sinh\left[\nu_0(y + H)\right]dy,$$

where

$$c_0 = \int_{-H}^{0} \sinh^2\left[\nu_0(y + H)\right]dy = \frac{H}{2}\left(\frac{\sinh(2\nu_0 H)}{2\nu_0 H} - 1\right). \tag{6.3}$$

Define now

$$v(x,y)=\begin{cases}\varphi_0(x,y) & |x|\leqslant x_0,\\ \varphi_0(x,y)-\phi_0(x)\sinh\bigl[\nu_0(y+H)\bigr] & |x|\geqslant x_0.\end{cases} \tag{6.4}$$

Then, $v\in U_*$ (see (4.7)) and by inserting it in the variational equation (3.5) we find, after suitable integrations by parts (see Theorem 6.2 below)

$$\int_S \nabla\psi\nabla\varphi_0\,dx\,dy=0. \tag{6.5}$$

By this equation we have that ψ is (weakly) harmonic for $x\neq\pm x_0$ in S; then, Eq. (6.1) is obtained by splitting the test functions space as a direct sum of the subspace of the functions satisfying (6.2) with its (two-dimensional) supplementary subspace. The boundary conditions now follow by the same arguments used in [9, Theorem 3.1]. Finally, it can be checked that $\hat\psi-\psi\in U_*$ and satisfies the variational equation (3.5) in U_*; then, we get $\hat\psi=\psi$ by coercivity. □

Theorem 6.2. *Assume that $\nu>1/b$ and let $\psi\in\mathcal{W}_{\mathcal{H}}$ be defined by Theorem* 5.1, *where $\mathcal{H}$ is given by* (5.5) *with arbitrary constants α, β. Then, ψ satisfies the boundary conditions of problem P_ν and there exist real constants λ_+, λ_-, μ_+, μ_- such that*

$$\begin{aligned}\Delta\psi(x,y)={}&\bigl[\lambda_+\delta(x-x_0)+\lambda_-\delta(x+x_0)\bigr]\sinh\bigl[\nu_0(y+H)\bigr]\\ &+\bigl[\mu_+\delta(x-x_0)+\mu_-\delta(x+x_0)\bigr]\sinh\bigl[\nu_1(y+b)\bigr]\chi_{[-b,0]}(y),\end{aligned} \tag{6.6}$$

for $(x,y)\in S$, *where $\chi_{[-b,0]}(y)$ denotes the characteristic function of the interval* $[-b,0]$. *Moreover, if $\hat\psi$ solves* (6.6) *with (possibly different) constants $\hat\lambda_\pm$, $\hat\mu_\pm$ and satisfies the same boundary conditions and the same additional condition* (5.4) *as ψ, then $\hat\psi=\psi$ (and therefore $\hat\lambda_\pm=\lambda_\pm$, $\hat\mu_\pm=\mu_\pm$).*

Proof. We provide a detailed proof of Eq. (6.6); then, the checking of the boundary conditions and of uniqueness will follow as in Theorem 6.1.

Let us now denote by φ_1 a smooth function with support in S, satisfying (6.2) and the additional conditions

$$\int_{-b}^{0}\varphi_1(\pm x_0,y)\sinh\bigl[\nu_1(y+b)\bigr]\,dy=0. \tag{6.7}$$

We further set

$$\phi_1(x)=c_1^{-1}\int_{-b}^{0}\varphi_1(x,y)\sinh\bigl[\nu_1(y+b)\bigr]\,dy,$$

where

$$c_1 = \int_{-b}^{0} \sinh^2\big[\nu_1(y+b)\big]\,dy = \frac{b}{2}\left(\frac{\sinh(2\nu_1 b)}{2\nu_1 b} - 1\right), \tag{6.8}$$

and define

$$v(x,y) = \begin{cases} \varphi_1(x,y) - \phi_1(x)\sinh[\nu_1(y+b)]\chi_{[-b,0]}(y), & |x| \leqslant x_0, \\ \varphi_1(x,y) - \phi_0(x)\sinh[\nu_0(y+H)], & |x| \geqslant x_0. \end{cases} \tag{6.9}$$

We readily check that $v \in V_*$ (see (5.6)); then we put (6.9) in the variational equation (3.5) taking account of (4.4), (5.4), (5.5) and integrate by parts the terms with the derivative ψ_y (see [9, Theorem 3.1] for an analogous calculation). By the boundary conditions and relations (4.2), (5.2) we find

$$\int_S \nabla\psi\nabla\varphi_1\,dx\,dy = \int_{-x_0}^{x_0} \big\{\phi_1'(x)\mathcal{H}'(x) - \phi_1(x)\nu_1\big[\mathcal{K}_+(x) + \nu_1\mathcal{H}(x)\big]\big\}\,dx.$$

Finally, by recalling that $\phi_1(\pm x_0) = 0$ we have

$$\int_S \nabla\psi\nabla\varphi_1\,dx\,dy = -\int_{-x_0}^{x_0} \big\{\phi_1(x)\big[\mathcal{H}''(x) + \nu_1^2\mathcal{H}(x) + \nu_1\mathcal{K}_+(x)\big]\big\}\,dx.$$

By (5.5) we readily verify that $\mathcal{H}''(x) + \nu_1^2\mathcal{H}(x) = -\nu_1\mathcal{K}_+(x)$; hence, the right-hand side vanishes for every φ_1 satisfying (6.2) and (6.7). Now (6.1) follows by a straightforward generalization of the argument of Theorem 6.1. □

7. Regularization and solvability of the problem

As we clarified in the previous section, a variational solution is not in general a solution of problem P_ν, since it is not harmonic in S. Roughly speaking, this is because the weak solution does not include the "oscillating parts" of the stream function which appear downstream the beam for $\nu > 1/H$ (with wave number given by (4.2)) and also between the beam and the free surface when $\nu > 1/b$ (with wave number given by (5.2)). In this section, we "regularize" the weak solutions described in Theorems 6.1 and 6.2 (respectively, in the two regimes $1/H < \nu < 1/b$ and $\nu > 1/b$) in such a way to obtain (unique) solutions of problem P_ν.

We first outline the case $1/H < \nu < 1/b$, as its regularization is very similar to the existing ones for a surface beam [4,12] and for a surface-piercing body [8]. The first step is to modify the weak solution ψ in order to obtain a function harmonic through S: let us set $\tilde{\Phi}_0(x,y) = \Phi_0(x,y;x_0)$ and $\tilde{\Psi}_0(x,y) = \Psi_0(x,y;x_0)$, where Φ_0, Ψ_0, are defined by (4.3). By elementary calculations, the function

$$\tilde{\psi} = \begin{cases} \psi + \frac{\lambda_-}{\nu_0}\tilde{\Phi}_0, & x < -x_0, \\ \psi, & |x| \leqslant x_0, \\ \psi - \frac{\lambda_+}{\nu_0}\tilde{\Psi}_0, & x > x_0, \end{cases} \tag{7.1}$$

is *harmonic in* S and satisfies the same boundary conditions as ψ; we remark, however, that $\tilde{\psi}$ is *not* a solution to problem P_ν (unless $\lambda_- = 0$) since it oscillates *both* at $+\infty$ and $-\infty$ and therefore does not satisfy the asymptotic condition (2.13). In order to get rid of the unwanted waves at $-\infty$, we exploit the properties of two variational solutions, denoted by ψ^s and ψ^c, satisfying special conditions on the beam: ψ^s and ψ^c are the two (uniquely determined) functions given by Theorem 4.1, where:

$$\text{for } \psi^s \quad \text{we take} \quad \mathcal{K}_\pm = \sin(\nu_0 x), \qquad \text{for } \psi^c \quad \text{we take} \quad \mathcal{K}_\pm = \cos(\nu_0 x). \tag{7.2}$$

By the symmetry properties of this data and by uniqueness, we have $\psi^s(-x, y) = -\psi^s(x, y)$ and $\psi^c(-x, y) = \psi^c(x, y)$; as a consequence, ψ^s and ψ^c satisfy (6.1) with $\lambda_+ = -\lambda_- \equiv \lambda^s$ and $\lambda_+ = \lambda_- \equiv \lambda^c$, respectively. For special values of the parameter ν_0, the functions $\psi^{s,c}$, and the coefficients $\lambda^{s,c}$ are explicitly known; for, by defining

$$\tilde{S}_0(x, y) = \frac{\sinh(\nu_0(y + H))}{\sinh[\nu_0(H - b)]} \sin(\nu_0 x), \qquad \tilde{C}_0(x, y) = \frac{\sinh(\nu_0(y + H))}{\sinh[\nu_0(H - b)]} \cos(\nu_0 x),$$

we have, for $\nu_0 x_0 = n\pi$, $n = 1, 2, \ldots,$

$$\psi^s(x, y) = \tilde{S}_0(x, y) \quad \text{for } |x| \leqslant x_0, \qquad \psi^s = 0 \quad \text{for } |x| > x_0, \tag{7.3}$$

with $\lambda^s = (-1)^{n+1}\nu_0/\sinh[\nu_0(H - b)]$. Similarly, if $\nu_0 x_0 = (n - 1/2)\pi$, we have

$$\psi^c(x, y) = \tilde{C}_0(x, y) \quad \text{for } |x| \leqslant x_0, \qquad \psi^c = 0 \quad \text{for } |x| > x_0, \tag{7.4}$$

with $\lambda^c = (-1)^{n+1}\nu_0/\sinh[\nu_0(H - b)]$.

Let us now modify ψ^s and ψ^c as in (7.1) with the correspondent coefficients λ^s, λ^c, in order to obtain functions $\tilde{\psi}^{s,c}$ harmonic through S. Finally, we set

$$\zeta_0^s = \tilde{\psi}^s - \tilde{S}_0, \qquad \zeta_0^c = \tilde{\psi}^c - \tilde{C}_0. \tag{7.5}$$

The harmonic functions ζ_0^s and ζ_0^c satisfy conditions (2.9)–(2.12) of problem P_ν *with vanishing boundary data*; furthermore, they have the same symmetry properties as ψ^s and ψ^c respectively, and oscillate at both directions at infinity. Hence, by a suitable combination of them, we may suppress the oscillations at $-\infty$ of $\tilde{\psi}$ in (7.1). Let us define

$$\omega = \tilde{\psi} + a\zeta_0^s + b\zeta_0^c \tag{7.6}$$

and choose the coefficients a and b in such a way that ω satisfies (2.13); then, ω will be the required solution of problem P_ν. By writing the asymptotic expression of ω for $x \to -\infty$ (which can be derived from (7.1) and (7.5) taking account that the variational parts, ψ and $\psi^{s,c}$, are exponentially vanishing) and imposing that the coefficients of the oscillatory terms vanish, we get the system

$$\begin{cases} \left[\lambda^s \cos(\nu_0 x_0) + \dfrac{\nu_0}{\sinh[\nu_0(H - b)]}\right] a - \lambda^c \cos(\nu_0 x_0) b = \lambda_- \cos(\nu_0 x_0), \\ \lambda^s \sin(\nu_0 x_0) a - \left[\lambda^c \sin(\nu_0 x_0) - \dfrac{\nu_0}{\sinh[\nu_0(H - b)]}\right] b = \lambda_- \sin(\nu_0 x_0). \end{cases} \tag{7.7}$$

This gives

$$a = \frac{\lambda_-}{\Delta}\cos(\nu_0 x_0), \qquad b = \frac{\lambda_-}{\Delta}\sin(\nu_0 x_0), \tag{7.8}$$

where

$$\Delta = \frac{\nu_0}{\sinh[\nu_0(H-b)]} + \lambda^s\cos(\nu_0 x_0) - \lambda^c\sin(\nu_0 x_0). \tag{7.9}$$

With this choice, the asymptotic expression of ω for $x \to +\infty$ writes

$$\omega = \mathcal{O}\big(e^{-\mu_1|x|}\big) + AS_0 + BC_0, \tag{7.10}$$

where μ_1 is the least positive solution of $\tan(\mu H) = \frac{\mu}{\nu}$ and characterize the decay at infinity of the variational solution [12]. The coefficients A, B of the oscillating terms can be easily calculated from (7.6) and (7.8) by taking account of the definitions (7.1), (7.5). We remark that, in the special case $\nu_0 x_0 = n\pi$ ($\nu_0 x_0 = (n - 1/2)\pi$), the function ζ^s (ζ^c) identically vanishes. Moreover, we will show below that

$$\Delta = J_0 \frac{\sinh[\nu_0(H-b)]}{c_0}\sin(\nu_0 x_0)\cos(\nu_0 x_0), \tag{7.11}$$

where $J_0 > 0$ for every $\nu_0 > 0$, i.e. for every $\nu > 1/H$ (see also [12, Proposition 4.8]) and the constant c_0 was defined in (6.3); nevertheless, the solution (7.6) as well as the coefficients A and B *have well-defined limits for* $\nu_0 x_0 \to m\frac{\pi}{2}$. The calculations are the same as in the case of the beam [4, Proposition A.2] (see also [8, Section 2.2]). The conclusion of this discussion is:

Theorem 7.1. *For* $1/H < \nu < 1/b$*, the function* ω *given by* (7.6) *is the unique solution to problem* P_ν*; for* $x \to +\infty$ (7.10) *holds, where the coefficients* A *and* B *are determined by* (7.8), (7.9) *and by the asymptotic expressions of the functions* (7.1), (7.5).

The assertion of uniqueness follows exactly in the same way as it was done for the surface beam [4, Theorem 3.2].

We now discuss the case $\nu > 1/b$, whose regularization requires a more careful analysis. In fact, if ψ is a weak solution as in Theorem 6.2, we now see that the function $\tilde{\psi}$ defined by (7.1) is no longer harmonic but it satisfies

$$\Delta\tilde{\psi}(x,y) = \big[\mu_+\delta(x-x_0) + \mu_-\delta(x+x_0)\big]\sinh\big[\nu_1(y+b)\big]\chi_{[-b,0]}(y). \tag{7.12}$$

On the other hand, we know that a variational solution is not uniquely determined by the boundary data, but it depends on two arbitrary constants (see (5.4), (5.5)); clearly, the same is true for $\tilde{\psi}$. Thus, it is natural to ask whether such constants can be determined by requiring $\mu_\pm = 0$ in the above equation. To begin with, we still denote by ψ^s, ψ^c the two weak solutions, given by Theorem 5.1, which satisfy (7.2) and the additional condition (5.4) respectively in the form

$$\int_{-b}^{0} \psi^s(\xi,y)\sinh\big[\nu_1(y+b)\big]\,dy = \frac{\nu_1}{\nu_0^2-\nu_1^2}\sin(\nu_0\xi),$$

$$\int_{-b}^{0} \psi^c(\xi, y) \sinh\big[\nu_1(y+b)\big]\, dy = \frac{\nu_1}{\nu_0^2 - \nu_1^2} \cos(\nu_0 \xi), \tag{7.13}$$

for every $\xi \in [-x_0, x_0]$. We remark that the above right-hand sides are obtained from (5.5) by inserting $\mathcal{K}_\pm$ given by (7.2) and with a suitable choice of the coefficients α, β. It is readily verified that ψ^s and ψ^c maintain the previous symmetry properties with respect to x; hence, they satisfy (6.6) respectively with $\lambda_+ = -\lambda_- \equiv \lambda^s$, $\mu_+ = -\mu_- \equiv \mu^s$ and $\lambda_+ = \lambda_- \equiv \lambda^c$, $\mu_+ = \mu_- \equiv \mu^c$. Moreover, by direct computations it follows that (7.3), (7.4) still hold respectively for $\nu_0 x_0 = n\pi$ and $\nu_0 x_0 = (n - 1/2)\pi$, with the same values of $\lambda^{s,c}$ and with $\mu^{s,c} = 0$.

By recalling Remark 5.2, we can now consider *another pair of special solutions of the variational problem*, denoted by χ^s, χ^c, with homogeneous boundary data and such that

$$\int_{-b}^{0} \chi^s(\xi, y) \sinh\big[\nu_1(y+b)\big]\, dy = \sin(\nu_1 \xi),$$

$$\int_{-b}^{0} \chi^c(\xi, y) \sinh\big[\nu_1(y+b)\big]\, dy = \cos(\nu_1 \xi), \tag{7.14}$$

for every $\xi \in [-x_0, x_0]$. The following properties are easily verified: $\chi^s(-x, y) = -\chi^s(x, y)$, $\chi^c(-x, y) = \chi^c(x, y)$; then, χ^s and χ^c satisfy (6.6) with $\lambda_+ = -\lambda_- \equiv \xi^s$, $\mu_+ = -\mu_- \equiv \upsilon^s$ and $\lambda_+ = \lambda_- \equiv \xi^c$, $\mu_+ = \mu_- \equiv \upsilon^c$, respectively. For special values of the parameter ν_1, the functions $\chi^{s,c}$, and the coefficients $\xi^{s,c}$, $\upsilon^{s,c}$ are explicitly known; in fact, by setting

$$\tilde{S}_1(x, y) = \frac{\sinh(\nu_1(y+b))}{c_1} \sin(\nu_1 x), \qquad \tilde{C}_1(x, y) = \frac{\sinh(\nu_1(y+b))}{c_1} \cos(\nu_1 x),$$

where the constant c_1 was defined in (6.8), we have for $\nu_1 x_0 = n\pi$, $n = 1, 2, \ldots,$

$$\chi^s(x, y) = \tilde{S}_1(x, y)\chi_{[-b,0]}(y) \quad \text{for } |x| \leqslant x_0, \qquad \chi^s = 0 \quad \text{for } |x| > x_0, \tag{7.15}$$

with $\xi^s = 0$ and $\upsilon^s = (-1)^{n+1}\nu_1/c_1$. Similarly, if $\nu_1 x_0 = (n - 1/2)\pi$, we have

$$\chi^c(x, y) = \tilde{C}_1(x, y)\chi_{[-b,0]}(y) \quad \text{for } |x| \leqslant x_0, \qquad \chi^c = 0 \quad \text{for } |x| > x_0, \tag{7.16}$$

with $\xi^c = 0$ and $\upsilon^c = (-1)^{n+1}\nu_1/c_1$.

Let us now define as before in (7.1) the functions $\tilde{\psi}^s$, $\tilde{\psi}^c$, $\tilde{\chi}^s$, $\tilde{\chi}^c$; furthermore, let ζ_0^s, ζ_0^c be given as in (7.5) and set

$$\zeta_1^s = \tilde{\chi}^s, \qquad \zeta_1^c = \tilde{\chi}^c.$$

Clearly, ζ_0^s, ζ_0^c and ζ_1^s, ζ_1^c satisfy the homogeneous boundary conditions of problem P_ν and Eq. (7.12) respectively with $\mu_\pm = \pm\mu^s$, $\mu_\pm = \mu^c$ and $\mu_\pm = \pm\upsilon^s$, $\mu_\pm = \upsilon^c$. Then, if ψ is a weak solution given by Theorem 6.2 and $\tilde{\psi}$ is defined as in (7.1), we set

$$\omega = \tilde{\psi} + a_0\zeta_0^s + b_0\zeta_0^c + a_1\zeta_1^s + b_1\zeta_1^c. \tag{7.17}$$

We want to choose the four constants a_0, a_1, b_0, b_1, in such a way to satisfy the *two* requirements:

- ω is harmonic in S,
- ω satisfies the asymptotic condition (2.13).

In that case, the function ω will be a solution to problem P_ν. Let us now write explicitly the two conditions above; to this aim, we define the following coefficients:

$$\begin{gathered}
\Lambda_c^s = \lambda^s \cos(\nu_0 x_0) + \frac{\nu_0}{\sinh[\nu_0(H-b)]}, \qquad \Lambda_c^c = -\lambda^c \cos(\nu_0 x_0), \\
\Lambda_s^s = \lambda^s \sin(\nu_0 x_0), \qquad \Lambda_s^c = -\lambda^c \sin(\nu_0 x_0) + \frac{\nu_0}{\sinh[\nu_0(H-b)]}, \\
\Xi_c^s = \xi^s \cos(\nu_0 x_0), \qquad \Xi_c^c = -\xi^c \cos(\nu_0 x_0), \\
\Xi_s^s = \xi^s \sin(\nu_0 x_0), \qquad \Xi_s^c = -\xi^c \sin(\nu_0 x_0).
\end{gathered} \tag{7.18}$$

Then, by explicit calculations and taking account of the asymptotic expression of ω as in (7.7), we get

$$\begin{cases}
\mu^s a_0 + \mu^c b_0 + \upsilon^s a_1 + \upsilon^c b_1 = -\mu_+, \\
-\mu^s a_0 + \mu^c b_0 - \upsilon^s a_1 + \upsilon^c b_1 = -\mu_-, \\
\Lambda_c^s a_0 + \Lambda_c^c b_0 + \Xi_c^s a_1 + \Xi_c^c b_1 = \lambda_- \cos(\nu_0 x_0), \\
\Lambda_s^s a_0 + \Lambda_s^c b_0 + \Xi_s^s a_1 + \Xi_s^c b_1 = \lambda_- \sin(\nu_0 x_0).
\end{cases} \tag{7.19}$$

This system is uniquely solvable if and only if

$$\begin{vmatrix}
\mu^s & \mu^c & \upsilon^s & \upsilon^c \\
-\mu^s & \mu^c & -\upsilon^s & \upsilon^c \\
\Lambda_c^s & \Lambda_c^c & \Xi_c^s & \Xi_c^c \\
\Lambda_s^s & \Lambda_s^c & \Xi_s^s & \Xi_s^c
\end{vmatrix} \neq 0.$$

By elementary calculations and by the definitions (7.18), we get

$$\mu^c \upsilon^s \xi^c \sin(\nu_0 x_0) - \mu^s \upsilon^c \xi^s \cos(\nu_0 x_0) + \upsilon^s \upsilon^c \Delta \neq 0, \tag{7.20}$$

where Δ is given by (7.9). In order to write condition (7.20) in a more explicit form, we exploit some features of the coefficients at the left-hand side:

Proposition 7.2. *The following relations hold*

$$\begin{gathered}
\mu^s = -I^s \sin(\nu_0 x_0), \qquad \mu^c = -I^c \cos(\nu_0 x_0), \\
\xi^s = I^s \frac{\sinh[\nu_0(H-b)]}{c_0} \sin(\nu_1 x_0), \qquad \xi^c = I^c \frac{\sinh[\nu_0(H-b)]}{c_0} \cos(\nu_1 x_0), \\
\upsilon^s = -\frac{\nu_1}{c_1} \cos(\nu_1 x_0) - J_b^s \sin(\nu_1 x_0), \qquad \upsilon^c = \frac{\nu_1}{c_1} \sin(\nu_1 x_0) - J_b^c \cos(\nu_1 x_0),
\end{gathered}$$

where $J^s > J^c > 0$, $J_b^s > J_b^c > 0$, $(I^s)^2 \leqslant J^s J_b^s$, $(I^c)^2 \leqslant J^c J_b^c$; the quantities J^s, J^c, J_b^s, J_b^c are quadratic functionals respectively of ψ^s, ψ^c, χ^s, χ^c, while I^s, I^c are quadratic functionals of the arguments ψ^s, χ^s and ψ^c, χ^c. Moreover, the coefficient Δ satisfies (7.11) *with $J_0 = J^s - J^c$. All the above terms depend analytically on ν.*

The proof is given in Appendix A; we only remark here that the above expressions are compatible with the values previously calculated from the explicit solutions (7.3), (7.4) and (7.15), (7.16). By the previous proposition and by elementary calculations, we can write (7.20) in the form

$$\sin(\nu_0 x_0)\cos(\nu_0 x_0)\big[\mathcal{P}\cos^2(\nu_1 x_0) + \mathcal{Q}\sin^2(\nu_1 x_0) + \mathcal{R}\sin(\nu_1 x_0)\cos(\nu_1 x_0)\big] \neq 0, \quad (7.21)$$

where

$$\mathcal{P} = \frac{\nu_1}{c_1}\big[\big(I^c\big)^2 + J_0 J_b^c\big], \qquad \mathcal{Q} = \frac{\nu_1}{c_1}\big[\big(I^s\big)^2 - J_0 J_b^s\big],$$
$$\mathcal{R} = \big(I^c\big)^2 J_b^s - \big(I^s\big)^2 J_b^c + J_0\left[J_b^s J_b^c - \left(\frac{\nu_1}{c_1}\right)^2\right].$$

Thus, the system (7.19) is uniquely solvable if $\nu_0 x_0 \neq k\pi/2$ ($k \in \mathbb{N}$) and whenever the factor in the square brackets of (7.21) does not vanish. Unfortunately, it is not clear how to write the latter condition in an explicit form; moreover, one should check whether the solution (7.17) could also be defined by a suitable limit procedure (as described before Theorem 7.1) when ν_0 (ν_1) approaches the roots of the left-hand side of (7.21). We will not pursue here this analysis, but we observe that the left-hand side of (7.21) is an analytic function (of the parameter ν) which is not identically vanishing (take ν such that $\nu_1 x_0 = n\pi$); thus, by recalling Theorem 7.1 we get:

Theorem 7.3. *There is a unique solution ω to problem P_ν for every value of ν, with the possible exception of a sequence of singular values $\nu^{(n)} \to +\infty$, with $\nu^{(n)} > 1/b$ for every n. For $x \to +\infty$ the solution satisfies* (7.10).

8. Final remarks and open problems

We have studied a plane problem for the stream function of a linearized flow around a "thin" submerged obstacle in a heavy ideal fluid. We proved unique solvability for every value of the unperturbed flow velocity c with the possible exception of a discrete set in the interval $0 < c < \sqrt{gb}$, where b is the obstacle's depth. This set must be a subset of the sequence $\{c_n\}_{n=1}^{\infty}$, $c_n \to 0$, where $\nu^{(n)} = g/c_n^2$ are the zero points of the left-hand side of (7.21). We stress that the coefficients in (7.21) only depend on the geometric data of the problem, that is the half width of the beam x_0, its depth b and the depth H of the channel's bottom. It remains an open question to verify if a given c_n is a true singular value of the problem or if one can recover unique solvability by a suitable limiting procedure as in the case $\sqrt{gb} < c < \sqrt{gH}$.

We now discuss the determination of a velocity field from a solution of problem P_ν (see Section 2). We first note that if ψ_1 solves P_ν with boundary data $\mathcal{K}_\pm = 1$ in condition (2.10) then the corresponding velocity field satisfies the *homogeneous condition* $\partial_x \psi_1 = v_1 = 0$ on the beam. We conjecture that this solution represents a kind of vortex flow with nontrivial circulation around the beam; in this case, by linearity, we can uniquely determine the parameter γ in the boundary data (2.16) by prescribing the value of the right-hand side of (2.17). Actually, the conjecture is verified for a supercritical flow since in that case the circulation through a closed curve $\mathcal{C}$ surrounding the beam can be written

$$\int_I \psi_1\{(\partial_y\psi_1)_+ - (\partial_y\psi_1)_-\}\,dx = \int_S |\nabla\psi_1|^2\,dx\,dy - \nu\int_F |\psi_1|^2\,dx > 0,$$

where we used the boundary conditions and the divergence theorem. The strict positivity of the last term follows by $\nu H < 1$ and by the Hölder inequality applied to the identity $\psi_1(x,0) = \int_{-H}^0 \partial_y\psi_1(x,s)\,ds$, which holds for every $x \in \mathbb{R}$ since ψ_1 is constant along the beam. In the case $\nu H > 1$ we cannot apply the previous argument, but we expect that ψ_1 still has nontrivial circulation at least for the regular values of the parameter ν; if not, the (well-defined) harmonic conjugate ϕ_1 would be a nontrivial solution of the homogeneous problem (2.1)–(2.7). Finally, we consider the problem of determining the least singular solution. We recall that any weak solution of (2.9), (2.10) in a neighborhood of I has the form $\psi = \psi_0 + c_1\psi_+ + c_2\psi_-$, where ψ_0 is an H^2 function and $|\nabla\psi_\pm| \sim r_\pm^{-1/2}$, for $r_\pm \to 0$, r_+ (r_-) being the distance to the right (left) end point of the beam [16]. Now, if $\psi = \psi_1$ the above coefficients cannot both vanish; otherwise, the derivative $\partial_x\psi_1$ would be a solution of the homogeneous problem P_ν. Thus, if ψ solves P_ν we can choose γ in (2.16) such that (at least) one of the singularities of $\nabla\psi$ disappears.

Appendix A

Proof of Proposition 7.2. By the expression of $\Delta\psi^{s,c}$, $\Delta\chi^{s,c}$, we have

$$\begin{aligned}
\psi_x^{s,c}\big(x_0^+,y\big) - \psi_x^{s,c}\big(x_0^-,y\big) &= \lambda^{s,c}\sinh\big[\nu_0(y+H)\big] + \mu^{s,c}\sinh\big[\nu_1(y+b)\big]\chi_{[-b,0]}(y),\\
\chi_x^{s,c}\big(x_0^+,y\big) - \chi_x^{s,c}\big(x_0^-,y\big) &= \xi^{s,c}\sinh\big[\nu_0(y+H)\big] + \upsilon^{s,c}\sinh\big[\nu_1(y+b)\big]\chi_{[-b,0]}(y).
\end{aligned}\tag{A.1}$$

Now, the explicit forms of $\psi^{s,c}$ and of $\chi^{s,c}$, respectively for $\nu_0 x_0 = n\pi/2$ and for $\nu_1 x_0 = n\pi/2$, $n = 1,2,\dots$ (see Section 7) justify the following definitions:

$$\begin{aligned}
\psi^s(x,y) &= \begin{cases} \sin(\nu_0x_0)\hat\psi^s(x,y) + \tilde S_0(x,y), & |x| \leqslant x_0,\\ \sin(\nu_0x_0)\check\psi^s(x,y), & |x| \geqslant x_0,\end{cases}\\
\psi^c(x,y) &= \begin{cases} \cos(\nu_0x_0)\hat\psi^s(x,y) + \tilde C_0(x,y), & |x| \leqslant x_0,\\ \cos(\nu_0x_0)\check\psi^s(x,y), & |x| \geqslant x_0,\end{cases}\\
\chi^s(x,y) &= \begin{cases} \sin(\nu_1x_0)\hat\chi^s(x,y) + \tilde S_1(x,y)\chi_{[-b,0]}(y), & |x| \leqslant x_0,\\ \sin(\nu_1x_0)\check\chi^s(x,y), & |x| \geqslant x_0,\end{cases}\\
\chi^c(x,y) &= \begin{cases} \cos(\nu_1x_0)\hat\chi^c(x,y) + \tilde C_1(x,y)\chi_{[-b,0]}(y), & |x| \leqslant x_0,\\ \cos(\nu_1x_0)\check\chi^c(x,y), & |x| \geqslant x_0,\end{cases}
\end{aligned}\tag{A.2}$$

where $\tilde S_0$, $\tilde C_0$, $\tilde S_1$, $\tilde C_1$, were defined in Section 7. Clearly, the functions at the right-hand sides of (A.2) are harmonic in S_{x_0} and in $S \setminus S_{x_0}$ respectively; moreover, they satisfy homogeneous boundary conditions on F, B and I. Finally, $\check\psi^{s,c}(x,y)$ and $\check\chi^{s,c}(x,y)$ verify (4.4), while the functions $\hat\psi^{s,c}(x,y)$, $\hat\chi^{s,c}(x,y)$ satisfy the *homogeneous condition* (5.4), that is,

$$\int_{-b}^{0} \hat{\psi}^{s,c}(x,y)\sinh\bigl[\nu_1(y+b)\bigr]\,dy = 0,$$
$$\int_{-b}^{0} \hat{\chi}^{s,c}(x,y)\sinh\bigl[\nu_1(y+b)\bigr]\,dy = 0, \tag{A.3}$$

for $|x| \leqslant x_0$. In fact, the first line is obtained by (7.13) and by the identity

$$\int_{-b}^{0} \sinh\bigl[\nu_0(y+H)\bigr]\sinh\bigl[\nu_1(y+b)\bigr]\,dy = \sinh\bigl[\nu_0(H-b)\bigr]\frac{\nu_1}{\nu_0^2-\nu_1^2},$$

which follows by explicit calculations using (4.2) and (5.2). The second equation follows from (7.14) and by the definition of c_1.

Then, by (4.4), (7.13) and (7.14), we get the following relations

$$\int_{-H}^{0} \bigl[\psi_x^{s,c}\bigl(x_0^+,y\bigr) - \psi_x^{s,c}\bigl(x_0^-,y\bigr)\bigr]\hat{\psi}^{s,c}(x_0,y)\,dy = -\frac{c_0}{\sinh[\nu_0(H-b)]}\lambda^{s,c}.$$

We now transform the left-hand sides taking into account (A.2), (A.3) and integrating by parts:

$$\int_{-H}^{0} \bigl[\psi_x^{s}\bigl(x_0^+,y\bigr) - \psi_x^{s}\bigl(x_0^-,y\bigr)\bigr]\hat{\psi}^{s}(x_0,y)\,dy = \frac{\nu_0 c_0}{\sinh^2[\nu_0(H-b)]}\cos(\nu_0 x_0) - J^s\sin(\nu_0 x_0),$$

where

$$J^s = -\Biggl[\int_{-H}^{0} \check{\psi}_x^{s}\bigl(x_0^+,y\bigr)\check{\psi}^{s}(x_0,y)\,dy - \int_{-H}^{0} \bigl[\hat{\psi}_x^{s}\bigl(x_0^-,y\bigr)\bigr]\hat{\psi}^{s}(x_0,y)\,dy\Biggr]$$
$$= \frac{1}{2}\Biggl[\int_{S\setminus S x_0} \bigl|\nabla\check{\psi}^s\bigr|^2\,dx\,dy - \nu\int_{|x|\geqslant x_0} \bigl|\check{\psi}^s\bigr|^2\,dx + \int_{S x_0} \bigl|\nabla\hat{\psi}^s\bigr|^2\,dx\,dy - \nu\int_{-x_0}^{x_0} \bigl|\hat{\psi}^s\bigr|^2\,dx\Biggr].$$

Similarly,

$$\int_{-H}^{0} \bigl[\psi_x^{c}\bigl(x_0^+,y\bigr) - \psi_x^{c}\bigl(x_0^-,y\bigr)\bigr]\hat{\psi}^{c}(x_0,y)\,dy = -\frac{\nu_0 c_0}{\sinh^2[\nu_0(H-b)]}\sin(\nu_0 x_0) - J^c\cos(\nu_0 x_0),$$

where

$$J^c = \frac{1}{2}\Biggl[\int_{S\setminus S x_0} \bigl|\nabla\check{\psi}^c\bigr|^2\,dx\,dy - \nu\int_{|x|\geqslant x_0} \bigl|\check{\psi}^c\bigr|^2\,dx + \int_{S x_0} \bigl|\nabla\hat{\psi}^c\bigr|^2\,dx\,dy - \nu\int_{-x_0}^{x_0} \bigl|\hat{\psi}^c\bigr|^2\,dx\Biggr].$$

By coercivity and by the Dirichlet principle we have (see [8, Lemma 2.4]) $J^s > J^c > 0$. Moreover,

$$\int_{-H}^{0} \left[\chi_x^s\left(x_0^+, y\right) - \chi_x^s\left(x_0^-, y\right)\right]\check{\chi}^s(x_0, y)\, dy = -\frac{\nu_1}{c_1}\cos(\nu_1 x_0) - J_b^s \sin(\nu_1 x_0),$$

$$\int_{-H}^{0} \left[\chi_x^c\left(x_0^+, y\right) - \chi_x^c\left(x_0^-, y\right)\right]\check{\chi}^c(x_0, y)\, dy = \frac{\nu_1}{c_1}\sin(\nu_1 x_0) - J_b^c \cos(\nu_1 x_0),$$

where

$$J_b^{s,c} = \frac{1}{2}\left[\int_{S\setminus Sx_0} \left|\nabla\check{\chi}^{s,c}\right|^2 dx\, dy - \nu \int_{|x|\geqslant x_0} \left|\check{\chi}^{s,c}\right|^2 dx + \int_{Sx_0} \left|\nabla\hat{\chi}^{s,c}\right|^2 dx\, dy - \nu \int_{-x_0}^{x_0} \left|\hat{\chi}^{s,c}\right|^2 dx\right].$$

We still have the inequalities $J_b^s > J_b^c > 0$. Furthermore,

$$\int_{-H}^{0} \left[\psi_x^s\left(x_0^+, y\right) - \psi_x^s\left(x_0^-, y\right)\right]\check{\chi}^s(x_0, y)\, dy = -I^s \sin(\nu_0 x_0),$$

$$\int_{-H}^{0} \left[\psi_x^c\left(x_0^+, y\right) - \psi_x^c\left(x_0^-, y\right)\right]\check{\chi}^c(x_0, y)\, dy = -I^c \cos(\nu_0 x_0),$$

where

$$\begin{aligned} I^{s,c} = {} & \frac{1}{2}\left[\int_{S\setminus Sx_0} \nabla\check{\psi}^{s,c}\cdot\nabla\check{\chi}^{s,c}\, dx\, dy - \nu \int_{|x|\geqslant x_0} \check{\psi}^{s,c}\check{\chi}^{s,c}\, dx\right] \\ & + \frac{1}{2}\left[\int_{Sx_0} \nabla\hat{\psi}^{s,c}\nabla\hat{\chi}^{s,c}\, dx\, dy - \nu \int_{-x_0}^{x_0} \hat{\psi}^{s,c}\hat{\chi}^{s,c}\, dx\right]. \end{aligned}$$

Similarly,

$$\int_{-H}^{0} \left[\chi_x^s\left(x_0^+, y\right) - \chi_x^s\left(x_0^-, y\right)\right]\hat{\psi}^s(x_0, y)\, dy = -I^s \sin(\nu_1 x_0),$$

$$\int_{-H}^{0} \left[\chi_x^c\left(x_0^+, y\right) - \chi_x^c\left(x_0^-, y\right)\right]\hat{\psi}^c(x_0, y)\, dy = -I^c \cos(\nu_1 x_0).$$

Finally, the bounds $(I^s)^2 \leqslant J^s J_b^s$, $(I^c)^2 \leqslant J^c J_b^c$ follow by the positivity of the previous quadratic forms and by elementary calculations. $\square$

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